COMBINED METHOD LEAST SQUARES - Another example

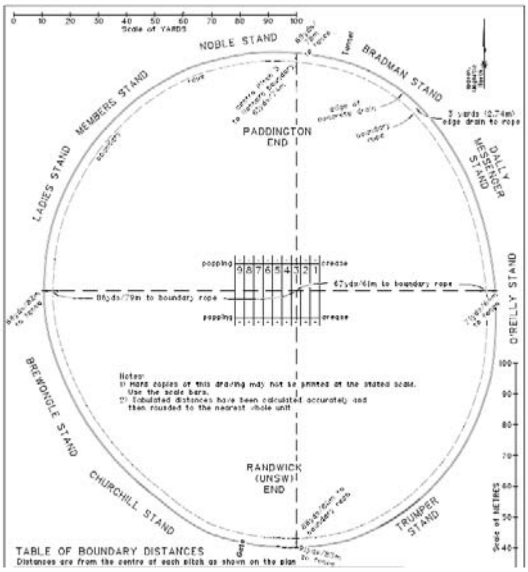
Question 6 in Chapter 8 of Monograph 13 provides coordinates around the UNSW Village Green (Cricket Oval) fence. The task is to find out if the sport field is an oval or a circle, by calculating a Least Squares solution for the radius and centre of the best fit circle. Also plot the residuals (orthogonal offsets) of the points from the circle.

The example (and additional question) in these notes applies to the Sydney Cricket Ground (SCG).

Background information:

Cricket ovals around the world can all be different sizes and shapes. The ICC Cricket standard dimensions of ovals for Test match, One-Day International and 20-20 games are: minimum 140 yards (130 m) from boundary to boundary square of the pitch. The straight boundary at both ends of the pitch shall be minimum 60 yards (55 m) from the centre of the pitch. So the fields do not have to be circles or even perfect ellipses. The minimum dimensions have the fields wider than long. Similarly, the field dimensions for Australian Rules Football can also vary from site to site. They are of oval shape, between 135 and 185 m in length and 110 and 155 m in width.

One of our UNSW graduates, David Barrington, has conducted many surveys at the SCG during its recent construction and changes. He has kindly supplied data for our example. From his large data set we selected 36 points approximately evenly spread around the edge of the oval. Part of one of his plans is shown below at right.



Our Task:

Is the SCG boundary close to a circle or an ellipse? From the photo and plan above we can clearly see the boundary is not a perfect geometric oval. So what are its best fit dimensions and how far does the boundary deviate from circle or ellipse? In these notes we will demonstrate the least squares estimation of a best fit circle and of a best fit ellipse, using the 36 points supplied below. Students are then invited to repeat the calculations using just half the data. Some students should use the odd numbered points and the other students the even numbered points.

Data:

The surveyed points around the oval, in a local plane coordinate system (with North aligned close to MGA North) are:

Point	E	N
1	850.459	1515.192
2	865.336	1513.154
3	874.081	1510.587
4	887.559	1504.106
5	898.823	1494.251
6	904.820	1486.189
7	910.295	1475.504
8	915.112	1461.306
9	916.848	1451.390
10	917.762	1441.396
11	917.492	1426.279
12	916.066	1416.320
13	912.583	1401.661
14	908.054	1391.617
15	900.549	1379.774
16	890.838	1369.650
17	879.249	1361.750
18	865.355	1356.077

Point	E	N
19	850.502	1354.145
20	835.540	1355.813
21	821.378	1360.225
22	808.740	1368.009
23	798.925	1378.053
24	788.301	1391.358
25	783.199	1400.037
26	777.364	1413.876
27	773.713	1428.488
28	772.737	1439.492
29	773.601	1449.425
30	778.961	1468.838
31	783.575	1477.749
32	792.577	1489.887
33	799.742	1496.918
34	811.946	1505.630
35	825.721	1511.878
36	835.440	1514.254

1. Circle fit

In Monograph 13, section 8.5.2 Volume of Water Tower, an example of a combined least squares circle fit is given. In this SCG example we don't have a measurement of circumference or tangent angle, just points around the circle.

Our equation that links the observations and parameters is:

$$0 = [(E_c - E_i)^2 + (N_c - N_i)^2] - r^2$$

This model requires a combined least squares solution. Our starting values for the parameters could simply be, for Ec the mean E of the points around the circle, and similarly for Nc. The starting value for r could be the distance from Ec Nc to any point. We choose the following starting values:

848.4 1437.8 77.4

For the input standard deviations of the observations we set ±0.03 m for all coordinates and assume no correlations.

There is one equation per point, so the b vector has 36 rows, the first five rows are:

b values
-2.928
25.748
33.374
60.853
261.600

There are 36 rows and 3 columns in the A matrix, the first five rows are:

A = df/dx		
/radius	/E c	/N c
-154.80	-4.12	-154.78
-154.80	-33.87	-150.71
-154.80	-51.36	-145.57
-154.80	-78.32	-132.61
-154.80	-100.85	-112.90

There are 36 rows and 72 (=2*36) columns in the B matrix, the top LH corner of B is:

	/e1	/n1	/e2	/n2	/e3	/n3	/e4	/n4	/e5	/n5
Pt 1	4.12	154.78	0	0	0	0	0	0	0	0
Pt 2	0	0	33.87	150.71	0	0	0	0	0	0
Pt 3	0	0	0	0	51.36	145.57	0	0	0	0
Pt 4	0	0	0	0	0	0	78.32	132.61	0	0
Pt 5	0	0	0	0	0	0	0	0	100.85	112.90

The combined least squares matrix calculations then yield:

Δx
-1.1156
-1.8426
-1.9337

We add these corrections to the starting values of r, Ec and Nc respectively, and continuing iterating this process until the Δx are all less than 1mm. The final results (rounded here to the nearest 0.1m) are: r = 76.3, Ec = 846.7, Nc = 1435.8. The best fit radius is about 4 m larger than the radius of the UNSW Village Green. How well does the data fit a circle? A simple Excel plot is shown below (remember Excel charts don't constrain both axes to the same scale, so there might be a smaller stretch in one axis). From the plot it is clear that a circle is not a good fit and that an ellipse that has a major axis NS longer than its EW axis might be a better fit to the data.

Residuals (v) and VF are not so relevant for this problem because the distances from points to the circle are much larger than the precision of the coordinates. More useful than v and VF for this problem are the distances between the points and the best fit circle. Those values can easily be calculated, Di = distance from point I to best fit centre minus best fit radius. A graph of the Di values is given below. It shows that the points can be up to about 6 metres from the circle, and the systematic pattern shows that none of the points is likely to be a gross error at the metre level.

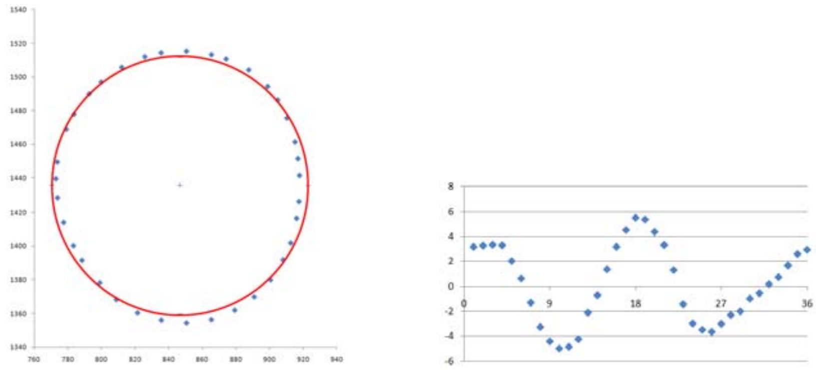


Fig. Plan of points with LS circle and graph of Distance (Di) from circle.

2. Ellipse (Oval) fit

Here we will solve for a best fit NS ellipse. So we will not try to solve for the bearing of the major axis. The reasons for that are based on the plan of the points with the best fit circle. It appears visually that a best fit ellipse would have a major axis with a bearing of close to 0 and that the length of the minor axis wouldn't be much shorter than the major axis. When the two axes are of similar length it is difficult to determine the bearing precisely.

There are several different ways to write the equation of an ellipse. We choose the following form. For each point there are two equations:

Ei = Ec + RA*Sin(brg + ti)
Ni = Nc + RA*Cos(brg + ti)

where

RA = (Maj * Min) / Sqr((Maj * sinti)^2 + (Min * costi)^2)
Ec Nc are coordinates of the centre
Maj Min brg are the semi major, semi minor and bearing of major axis
ti are the clockwise angles between the major axis the line from the centre to the point,

Since there is one equation for each observation we can use the Parametric Method.

The parameters are the centre coordinates (Ec, Nc) major and minor axis length, but not bearing of major axis, and the t value for each point.

With n points:
Number of observations = 2*n
Number of equations = 2*n
Number of parameters = 4 + n

Obs, in order: e1 n1 e2 n2 e3 n3 e4 n4 e5 n5 ...

Parameters, in order: Ec Nc (centre of ellipse) maj min brg t1 t2 t3...

The partial derivatives for the A matrix can be found by differentiating the above equations.

Partial E/maj:
RA*sin(brg+ti)*((1/maj)-(mag*sinti*sinti/((mag*sinti)^2+(min*costi)^2))
Partial E/min:
RA*sin(brg+ti)*((1/min)-(min*costi*costi/((mag*sinti)^2+(min*costi)^2))
Partial N/maj:
RA*cos(brg+ti)*((1/maj)-(mag*sinti*sinti/((mag*sinti)^2+(min*costi)^2))
Partial N/min:
RA*cos(brg+ti)*((1/min)-(min*costi*costi/((mag*sinti)^2+(min*costi)^2))

Partial E/Ec = 1, Partial E/Nc = 0, Partial N/Ec = 0, Partial N/Nc = 1

Partial E1/t1 = RA*COS(brg+t1) Partial N1/t1 = -RA*SIN(brg+t1)
All other partials / t1 = 0

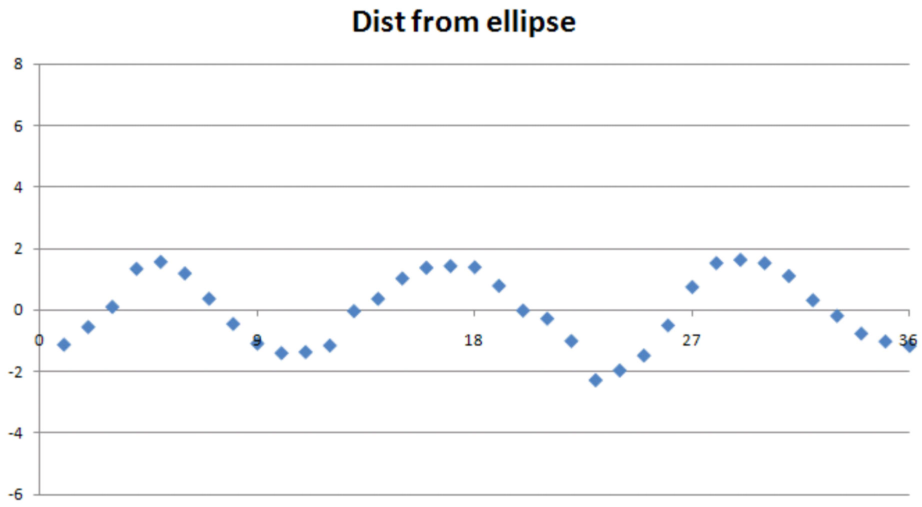
Starting values can be selected based on the best fit circle results.

After sufficient iterations, the adjusted values are:
Ec = 846.760
Nc = 1435.621
maj = 80.797
min = 72.584
brg fixed at zero i.e. NS ellipse

For the first few points the relevant (rounded) values are:

					A Matrix …										
Pt	obs	ti	RA	b vect	Ec	Nc	maj	min	t1	t2	t3	t4	t5	t6	etc
1E	850.5	2.7	80.8	-0.05	1	0	0.05	0.00	80.7	0	0	0	0	0	
1N	1515.2			-1.12	0	1	1.00	0.00	-3.8	0	0	0	0	0	
2	865.3	13.5	80.3	-0.13	1	0	0.22	0.02	0	78.1	0	0	0	0	
	1513.2			-0.54	0	1	0.90	0.07	0	-18.7	0	0	0	0	
3	874.1	20.0	79.7	0.04	1	0	0.29	0.05	0	0	74.9	0	0	0	
	1510.6			0.10	0	1	0.80	0.15	0	0	-27.3	0	0	0	
4	887.6	30.8	78.4	0.68	1	0	0.34	0.17	0	0	0	67.3	0	0	
	1504.1			1.15	0	1	0.58	0.28	0	0	0	-40.1	0	0	
	etc														

Similar to the circle fit, we can calculate the distances from the points to the best fit ellipse. The graph below shows that the ellipse fits the points better than the circle but there are still deviations of ±2 m.



Perhaps we can fit a higher order polynomial to the data? You are welcome to try. But before that you should attempt to replicate the above results for a circle and for a NS ellipse with 18 points of data.