

# Catenary Curves

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These notes are a worked example of least squares data analysis for students.

## Background

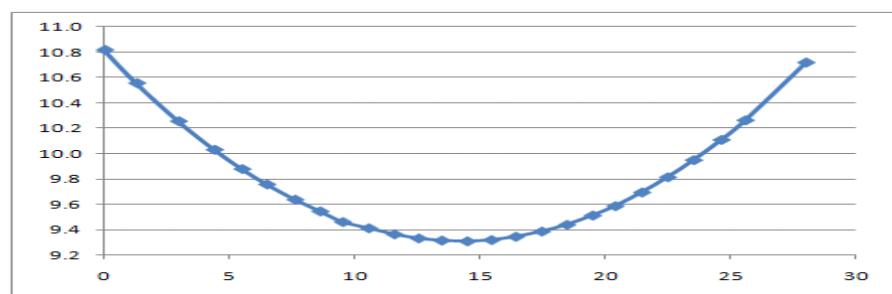
Some background reading about catenary curves will yield information such as the following. A freely hanging cable, such as a power line or unloaded flying fox, follows a catenary curve when supported at its ends and acted on only by its own weight. The catenary curve is based on the cosh (hyperbolic cosine) function, similar to a parabola. Apparently the fact that the curve followed by a cable or chain is not a parabola was proven by Jungius in 1669. Generally, the lighter the cable the more like a parabola it becomes. Most suspension bridge cables follow a parabolic not catenary curve, because they support the bridge deck below. An inverted catenary is an optimal shape for an arch.

At survey camps at Berry we sometimes measure the 350m flying fox cable and then fit a best fit curve to the data to determine the chainage and height of the lowest point on the cable, to investigate the residuals for each point compared to the best fit curve, and then to determine the clearance to the ground beneath the cable. In 2011 due to very wet weather we measured a 30m electrical extension cord hanging in an indoor basketball court. From the Sokkia reflector-less total station observations we determined height and chainage along the cable for about twenty points. Coordinates for three points are the minimum required to provide a unique solution with no redundancy.

## Data set

Chainage	Height
0.000	10.815
1.281	10.554
2.952	10.254
4.385	10.030
5.485	9.878
6.472	9.757
7.599	9.637
8.596	9.545
9.500	9.461
10.537	9.412
11.563	9.364
12.529	9.332
13.452	9.315
14.470	9.309
15.436	9.319
16.407	9.345
17.449	9.386
18.451	9.439
19.479	9.512
20.381	9.586
21.444	9.694
22.475	9.813
23.512	9.948
24.621	10.107
25.575	10.261
27.999	10.716

In 2011 one group observed 26 points along the centreline of the cable. The EN coordinates of their points were converted to chainage along the cord. Chainage can be thought of as the x axis and Height as the y axis of the data. This data will be used for worked examples of the various solutions presented below. **Note the graph below is not to scale.**



### Solution 1, spreadsheet polynomial trend line

I won't go into much detail for this method because there are better methods later. One way to fit a parabola in a MS Excel spreadsheet is to insert an xy scatter chart, click on it, then Chart Tools | Layout | Trendline | Polynomial | Order 2...

The result displayed is  $y = 0.0074x^2 - 0.2115x + 10.814$

Where x are the chainages, and y are the heights (all in metres).

If we substitute our chainage values into the equation the calculated y value can be compared with the measured heights. The differences are our residuals. If you do that with the data you will see that the residuals grow to about 20mm near the end of the curve even though we expect residuals of only a few mm. The reason is that the displayed coefficients in the equation do not reveal enough decimal places for our application. A better solution would be obtained using the DATA | ANALYSIS | DATA ANALYSIS process with MS Excel's Analysis Toolpack add in.

### Solution 2, parametric LS polynomial

Here we use the equation  $y = ax^2 + bx + c$  and rearrange it into our model form:  $0 = ax_i^2 + bx_i + c - y_i$

Where  $x_i$  are the observed chainages of the measured points,  $y_i$  are the observed heights of the measured points, and a b c are parameters. In this parametric model the x values are treated as "error free" constants and the y values as the only observation. I set the standard deviation of the y values for all points to  $\pm 1$ mm and treat the points as not being correlated. So the Q matrix, and thus the P matrix, are both diagonal. Next we form the (omc) b vector and the partial derivatives, A matrix. However to save computer space and the time taken for students to enter the zeros in the Q or P matrix we will not store or use Q or P. Instead we use the technique of dividing each term in b and each row in A by the standard deviation of that relevant observation. The Least Squares matrix calculations can then be simplified from  $N = A^T P A$  and  $dx = (A^T P A)^{-1} A^T P b$  by dividing the terms in A and b by  $s_i$ . So,  $N = (A/s)^T (A/s)$  and  $dx = N^{-1} (A/s)^T (b/s)$

To limit space I show the working for only the first three points. That should be enough for students to check their work as you process this data yourself.

The terms in b/s are calculated from  $=(y_i - (c + b * x_i + a(x_i)^2)) / s_i$

Using the values from solution 1 above as the starting values then the b/s vector for the first 3 points is:

1.000  
-1.212  
-0.138

The partial derivative of the model equation with respect to the parameter, a is  $x_i^2$

The partial derivative of the model equation with respect to the parameter, b is  $x_i$

The partial derivative of the model equation with respect to the parameter, c is 1

With parameters in the order a, b, c, the terms in the A matrix, divided by s, for the first three observations are:

0.0	0	1000
1641.0	1281	1000
8714.3	2952	1000

The  $\Delta x$  vector for the first iteration is

2.69E-05  
0.0000  
-0.0002

We add these to the starting values and iterate the solution until the  $\Delta x$  vector is close enough to zero for our purposes. This yields adjusted values of the parameters as: 0.0074269, -0.21149, 10.81378. These agree with our solution 1 but show more decimal places.

The Qx matrix ( $=N^{-1}$ ) is

1.09E-11   -3E-10   1.42E-09  
-3E-10   8.96E-09   -4.8E-08  
1.42E-09   -4.8E-08   3.45E-07

From that we can see the standard deviations of the parameters are: 0.0000033, 0.0001, 0.0006 and the correlations between the parameters are,  $\rho_{ab} = -0.96$ ,  $\rho_{ac} = 0.73$ ,  $\rho_{bc} = -0.86$ . The correlations are high. Our solution 1 method did not yield the standard deviations and correlations of the parameters.

Next we can calculate the residuals using  $v_i = -(y_i - (c + b \cdot x_i + a(x_i)^2))$ . This formula is similar to the formula for  $b_i$ , but has opposite sign and note that it uses the adjusted values of  $a$ ,  $b$  and  $c$ , not the starting values. The residuals for the first 3 observations are: -0.001, 0.001, 0.000 m. All the residuals are less than about 3mm, except for the one at chainage 9.5m which has a residual of 14mm. Either the chainage or height for that point is in error. We should look at the student's recorded data and their pre-processing to see if we can find the fault. However, here we will merely down-weight that observation with an input  $s$  of  $\pm 900$ mm. Recomputing the solution yields parameters and their standard deviations as  $a = 0.0074200 \pm 0.0000033$ ,  $b = -0.2113 \pm 0.0001$ , and  $c = 10.8140 \pm 0.0006$ . We divide the residuals  $v$  by their standard deviation  $s$ , and calculate the Variance Factor VF.  $VF = \Sigma(v/s)^2 / (n-u)$  where  $n = 26$  points and  $u = 3$  parameters. Our VF is 1.579.

If we wanted to know the chainage and height of the lowest point on the curve, we note that the lowest point on the parabolic curve is at  $dy/dx = 0$ . That is,  $x_L = -b/2a$  and  $y_L = ax_L^2 + bx_L + c$ . For this data  $x_L = 14.241$  and  $y_L = 9.309$  which seems reasonable considering the graph of  $xy$  data shown with the data. Actually, these coordinates are for the centre of the cable, so the bottom edge of the cable would be lower by size of the radius of the cable. Incidentally, note that from the graph of the original data it is not possible to see the residuals with the "naked eye". If we want the standard deviation of the  $x_L$  and  $y_L$  values we need to do propagation of variance calculations. Try that yourself if you use this method, otherwise the next method does it for you.

### Solution 3, parametric LS polynomial with lowest point parameters

The equation of a parabola can also be expressed as:  $Y_i = Y_L + A(X_i - X_L)^2$

where  $X_i$  is the horizontal 'coordinate' of a point (like chainage),  $Y_i$  is the height of the point,  $Y_L$  is the height of the lowest point on the curve,  $X_L$  is the  $x$  coordinate of the lowest point on the curve, and  $A$  is a coefficient describing the shape of the curve. So in this method the three parameters include the two coordinates of the lowest point on the curve, which is more meaningful and useful than the plain  $a$ ,  $b$ ,  $c$  coefficients of method 1 and 2 above. As starting values for the parameters we could look at the plot shown with the data, (let's use  $X_L = 14$  and  $Y_L = 9.3$ ) or simply select the point with the lowest observed height. The starting value for  $A$  could be based on our previous solutions or we could

calculate it as follows. If we select  $X_L = 14$  and  $Y_L = 9.3$  and then choose an observed point on the curve, say the first point ( $X_i=0, Y_i = 10.815$ ) then  $Y_i = Y_L + A(X_i - X_L)^2$  so  $A = (Y_i - Y_L) / (X_i - X_L)^2 = 0.0077$ .

Again,  $X_i$  are treated as constants and  $Y_i$  as observations to allow the parametric LS method. Also we divide  $b$  and  $A$  terms by the appropriate value of  $s_i$ . In this example we will continue with  $s = \pm 1\text{mm}$  for all points except at  $X_i = 9.5$  which will have  $s = \pm 900\text{mm}$  again to down-weight it. Let's use starting values of 0.007 for  $A$ ,  $X_L = 14$  and  $Y_L = 9.3$ .

The terms in  $b/s$  are calculated from  $= (Y_i - (Y_L + A(X_i - X_L)^2)) / s_i$

The first three terms are:

143.000  
121.589  
99.592

The partial derivative/s of the model equation with respect to the parameter,  $A$  is  $(X - X_L)^2 / s$

The partial derivative/s of the model equation with respect to the parameter,  $X_L$  is  $-2A(X - X_L) / s$

The partial derivative/s of the model equation with respect to the parameter,  $Y_L$  is  $1/s$

With parameters in the order  $A, Y_L, X_L$  the first three rows of  $A/s$  are:

196000.0	1000.	196.00
161773.0	1000.	178.07
122058.3	1000.	154.67

We iterate the solution until the  $\Delta x$  vector is close enough to zero for our purposes. This yields adjusted values of the parameters as:  $A = 0.007420$ ,  $Y_L = 9.3091$  and  $X_L = 14.2414$ . Their standard deviations (from the square roots of the diagonals of  $N^{-1}$  matrix) are: 0.000003, 0.0003, and 0.0017.

The residuals  $v$  are calculated from the adjusted values of the parameters inserted in the formula  $= -(Y_i - (Y_L + A(X_i - X_L)^2))$ . Then calculate  $v/s$  for each observation and the VF, as described in the previous method. The answers are the same as for the previous method, however this method gives us the values of  $X_L$  and  $Y_L$  and their standard deviations directly.

From the  $Qx$  matrix, the correlations between the parameters are,  $\rho_{AYL} = -0.72$ ,  $\rho_{AXL} = -0.13$ ,  $\rho_{YLXL} = -0.08$ . These correlations are low and much better than those in our solution 2 method. This method is thus more numerically stable as well as giving us more useful parameters directly.

#### **Solution 4, parametric LS catenary curve with lowest point parameters**

The equation of a catenary can be expressed in a few different forms. Here we use:

$$Y_i = Y_L + B(\cosh((X_i - X_L)/B) - 1)$$

where  $X_i$  is the horizontal 'coordinate' of a point (like chainage),  $Y_i$  is the height of the point,  $Y_L$  is the height of the lowest point on the curve,  $X_L$  is the  $x$  coordinate of the lowest point on the curve, and  $B$  is a coefficient describing the shape of the curve where a larger  $B$  indicates a flatter curve. Note that the coefficient  $B$  in this equation is usually considerably different to the coefficient  $A$  in the parabola equation. [Some surveyors may not have seen the cosh function since their time at university – it is available in spreadsheets.]

The starting value for B can be calculated as follows. If we select  $X_L = 14$  and  $Y_L = 9.3$  and then choose an observed point on the curve, say the first point ( $X_i=0, Y_i = 10.815$ ) then  $Y_i = Y_L + B(\cosh((X_i - X_L)/B) - 1)$  so  $B = 65$  (MS Excel's solver helps to find B).

If we assume that the catenary and parabola are very close then:  $B(\cosh(1/B) - 1) \approx A$

Again,  $X_i$  are treated as constants and  $Y_i$  as observations to allow the parametric LS method. Also we divide b and A terms by the appropriate value of  $s_i$ . In this example we will continue with  $s = \pm 1\text{mm}$  for all points except at  $X_i = 9.5$  which will have  $s = \pm 900\text{mm}$  again to down-weight it. Let's use starting values of 65 for B,  $X_L = 14$  and  $Y_L = 9.3$ .

The terms in  $b/s$  are calculated from  $= (Y_i - (Y_L + B(\cosh((X_i - X_L)/B) - 1)))/s_i$

The first three terms are:

1.470  
5.617  
12.827

The partial derivatives divided by  $s$ , of the model equation with respect to the parameters are:

$$(\partial F / \partial B) / s = (\text{COSH}((X_i - X_L) / B) - 1 - \text{SINH}((X_i - X_L) / B) * (X_i - X_L) / B) / s$$

$$(\partial F / \partial X_L) / s = -\text{SINH}((X_i - X_L) / B) / s$$

$$(\partial F / \partial Y_L) / s = 1 / s$$

With parameters in the order B,  $Y_L$ ,  $X_L$  the first three rows of  $A/s$  are:

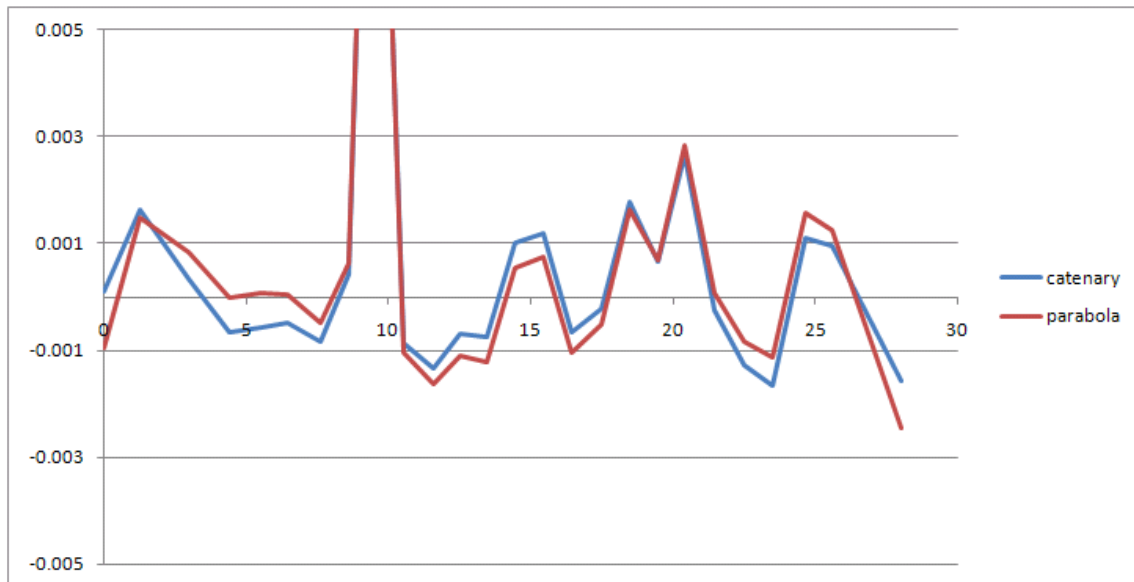
-23.465	1000	217.05
-19.328	1000	196.93
-14.549	1000	170.79

We iterate the solution until the  $\Delta x$  vector is close enough to zero for our purposes. This yields adjusted values of the parameters as:  $B = 67.60$ ,  $Y_L = 9.3096$  m and  $X_L = 14.2408$  m. Their standard deviations (from the square roots of the diagonals of  $N^{-1}$  matrix) are:  $\pm 0.03$ ,  $\pm 0.0003$  m height, and  $\pm 0.0017$  m chainage. Because the curve is flat we can determine the height of the lowest point more precisely than its chainage.

Note that the coordinates of the lowest point on the best fit catenary curve are very similar to those of the best fit parabola. The differences (0.5mm and 0.5mm in chainage and height) are less than the standard deviations of those parameters.

The residuals  $v$  are calculated from the adjusted values of the parameters inserted in the formula  $v_i = -(Y_i - (Y_L + B(\cosh((X_i - X_L)/B) - 1)))$ . Then calculate  $v/s$  for each observation and the VF, as described in the previous method. The largest  $v$  is 3mm (apart from the outlier at 9.5m chainage which has a residual of 15mm). The  $v$  and  $v/s$  are virtually the same as for the parabola methods, however this catenary method gives us a VF of 1.35 compared to the parabola's 1.58 which indicates that the catenary curve fits the data slightly better than the parabola and has slightly smaller residuals.

From the  $Q_x$  matrix, the correlations between the parameters are,  $\rho_{BYL} = 0.72$ ,  $\rho_{BXL} = 0.13$ ,  $\rho_{YLXL} = 0.08$ . These correlations are low and much better than those in our solution 2 method. This method is thus more numerically stable than the simple parabola with abc coefficients, as well as giving us more useful parameters directly.



Comparison of the residuals from the two types of curve fit

### Solution 5, combined method LS polynomial with lowest point parameters

In solution 3 we assumed the X values were error free, which allowed us to use the parametric method because each point generates one equation with only one observed value in it. Now we apply the combined method of Least Squares and treat both X and Y values as observations. We will also omit the outlier observation at chainage 9.5, so there are 25 observed points and 50 observations and we will use and input standard deviation of  $\pm 1\text{mm}$  for both chainage and height values.

The basic model equation is:  $Y_i - Y_L - A(X_i - X_L)^2 = 0$

The observations, in order are:  $Y_1, X_1, Y_2, X_2, \dots$

The parameters, in order are:  $A, Y_L, X_L$

The starting values of the parameters are:  $A = 0.0077, Y_L = 9.3$  and  $X_L = 14$ . (as in our previous solution 3).

Here we will not divide b and A terms by  $s_i$ , instead we will form the full Q matrix even though it is diagonal in this case. In this example we will continue with  $s = \pm 1\text{mm}$

The terms in b are calculated from  $= -F(X_a, l) = -(Y_i - Y_L - A(X_i - X_L)^2)$

The first three terms are:

-0.006

-0.008

-0.014

The partial derivative of the model equation with respect to the parameter, A is  $-(X - X_L)^2$

The partial derivative of the model equation with respect to the parameter,  $Y_L$  is -1

The partial derivative of the model equation with respect to the parameter,  $X_L$  is  $2A(X - X_L)$

With parameters in the order  $A, Y_L, X_L$  the first three rows of the A matrix, which has 25 rows and 3 columns, are:

-196.00 -1 -0.216  
 -161.77 -1 -0.196  
 -122.06 -1 -0.170

For the combined LS method we calculate a B matrix,  $B = \partial F / \partial \theta$ .

The derivative of the model equation with respect to a Y observation = 1

The derivative of the model equation for the first point with respect to observation  $X_1 = -2A(X_1 - X_L)$

Many of the derivatives are 0. The first three rows of the B matrix, which has 25 rows and 50 columns, are:

1	0.2156	0	0	0	0	0	0	0	...
0	0	1	0.1959	0	0	0	0	0	...
0	0	0	0	1	0.1701	0	0	0	...

The Q matrix has 50 rows and 50 columns. The diagonals are variances of the observations, in this example,  $0.001^2$ . The  $(BQB^T)^{-1}$  matrix has 25 rows and 25 columns, it can then be formed by matrix algebra. The top left hand corner of  $(BQB^T)^{-1}$  is:

955581.	0	0
0	963051.	0
0	0	971867.

The  $Q_x$  matrix has 3 rows and 3 columns, it can then be calculated from  $Q_x = (A^T(BQB^T)^{-1}A)^{-1}$ . For this first iteration  $Q_x$  is:

1.14E-11	-7.07E-10	-3.66E-10
-7.07E-10	8.42E-08	2.72E-08
-3.66E-10	2.72E-08	2.79E-06

The  $\Delta x$  vector is calculated from  $Q_x(A^T(BQB^T)^{-1}b)$  as:

-0.000280  
 0.0096  
 0.2326

We iterate the solution until the  $\Delta x$  vector is close enough to zero for our purposes. This yields adjusted values of the parameters as:  $A = 0.007420$ ,  $Y_L = 9.3092$  and  $X_L = 14.2414$ . Their standard deviations (from the square roots of the diagonals of  $Q_x$  matrix) are: 0.000003, 0.0003, and 0.0017. Both the adjusted values and their standard deviations are very close to the values from solution 3.

The residuals  $v$  in the combined method are calculated from the formula  $v = -QB^T(BQB^T)^{-1}(A\Delta x - b)$ . There are 50 observations, so there are 50 values in the  $v$  vector. If we use the last iteration then  $\Delta x$  are virtually zero and we use  $b$  from the last iteration. We can then calculate  $v/s$  for each observation and the VF. The first few  $v$  and  $v/s$  terms are:

Residuals	$v$	$v/s$
X1	-0.0009	-0.92
Y1	-0.0002	-0.19
X2	0.0014	1.44

$VF = v^T P v / (n_e - u) = 35.71 / (50 - 3) = 0.76$

Notice that the VF (and thus the residuals) is considerably smaller when we use the combined LS method, compared to the parametric method but the answers for the parameters are virtually the same for both methods (for this data set). This means that the data fits the curve better than we thought when we used the parametric method. The standard deviations of the observations are probably  $< \pm 1\text{mm}$ , and the standard deviations of the adjusted parameters from the combined method are thus also slightly better.

The interpretation of this best fit curve is similar to the difference between linear regression where the residuals for each point are vertical only, compared to a line of best fit where points can be corrected across as well as vertically.

### Solution 6, combined method LS catenary curve with lowest point parameters

This solution is not written here. It is left as an exercise for students to complete themselves. The partial derivatives in the B matrix need to be derived, the solution calculated, and the results compared with the previous solutions.

### Summary

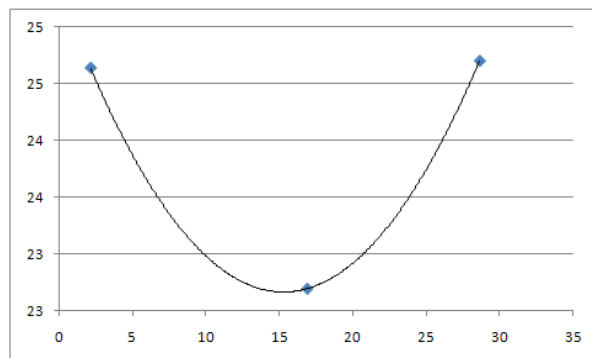
- At least for our data sets, it can be said that the parabola and catenary curves yield very similar results, with the catenary curve fitting the data slightly better than the parabola.
- I recommend you fit a curve with parameters being the coordinates of the lowest point on the curve and a parameter for shape, in preference to a simple second order polynomial.
- You don't have to hold any points fixed in these solutions.
- It is useful to calculate and analyse the residuals, and VF.
- It is useful to incorporate standard deviations of observations in a least squares adjustment of the data rather than simply treating each point as equal weight. That is because it allows us to down-weight low quality points and to determine realistic standard deviations or confidence intervals of the derived parameters.
- A combined LS method solution yields similar parameters (for this data set) but fits the data better with smaller residuals.
- Further work: It would be interesting to take 3D data for points along a cable, with the E, N, and H coordinates of each point being obtained by radiation with a total station or laser scanner and the coordinate observations being correlated. The best fit curve would also need to determine the bearing / azimuth of the cable.

### Tutorial Questions:

#### A) 3 point question

A subset of some student data is:

X	Y (Height)
2.134	24.637
16.876	22.694
28.624	24.698



Calculate the coordinates of the lowest



point on the curve if a catenary curve is fit to the data. Similarly, calculate the coordinates of the lowest point on the curve if a parabola is fit to the data. Compare the differences.

The solution can be found by using the Solver Function in MS Excel to set the residuals for all three data point's equations to zero. Alternatively, solve it by the least squares method.

**B) A different data set with one end of the cable considerably higher than the other**

C (Chainage)	H (Cable Height)
0.000	15.452
10.172	14.473
26.171	13.491
43.239	12.652
65.430	11.884
73.568	11.699
105.081	11.319
133.952	11.718
227.069	17.201
249.054	19.449
286.825	24.149
329.697	30.794
340.381	32.669

**C) 3D data**

Points along a cable were radiated by a reflectorless total station from a single point. Calculate the best fit curve by calculating the chainages of points from a best fit line in plan view and then a curve fit in elevation view. Alternatively, solve for the best fit curve in 3D. If you use a LS survey network program in simulation mode you can even calculate the correlations between the coordinates of the points.

Direction, D M S	Horizontal Distance	Height Difference
240 02 25	12.487	5.000
237 10 37	11.489	4.637
233 31 04	10.487	4.285
228 42 02	9.451	3.940
220 46 31	8.242	3.559
209 32 08	7.181	3.228
195 56 40	6.496	2.982
186 11 57	6.288	2.859
175 29 25	6.267	2.770
158 11 38	6.726	2.679
140 52 38	8.061	2.694
133 28 52	9.082	2.776
126 58 07	10.397	2.928
123 30 19	11.318	3.069
119 39 45	12.632	3.305
115 17 09	14.643	3.749
113 12 58	15.870	4.068
111 24 44	17.137	4.439
110 22 32	17.958	4.698